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ON THE PROBLEM OF A THREE-DIMENSIONAL CRACK IN AN ANISOTROPIC ELASTIC MEDIUM*

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A three-dimensional problem of elasticity theory for a homogeneous and anisotropic medium containing an isolated crack is considered. As a result of using a known analogy between a crack and a Somigliana dislocation, the problem is reduced to the solution of an elliptic pseudo-differential equation in the density of the dislocation moments simulating the crack. The properties of the operator in this equation are investigated. A regular representation of this operator is obtained in the class of sufficiently smooth functions. The possibility of applying the proposed regularization to a numerical solution of the problem is discussed. The structure of the tensor stress intensity factor on the smooth outline of an arbitrary crack in an anisotropic medium is analyzed.

1. Dislocation model of a crack. Let us consider an infinite homogeneous and anisotropic elastic medium in which there is a slit on a smooth bounded surface Ω (a three-dimensional crack). We shall consider the external stress field $\sigma_0(x)$ (the strain $\varepsilon_0(x)$) to be realized by loads applied at infinity, and the edges of the crack to be free of external loads ($x(x_1, x_2, x_3)$ is a point of the medium).

If the edges of the crack are not joined during loading of the medium, then the boundary condition on Ω has the form

$$n_\alpha(x)\sigma^{\alpha\beta}(x) = 0, \quad x \in \Omega \tag{1.1}$$

where $n(x)$ is the normal to the surface Ω and $\sigma(x)$ is the stress tensor.

In cases when the crack edges make contact, the boundary conditions on Ω will be more complex, depending on the nature of the edge interaction.

The displacement vector $u(x)$, which is a continuous function in all space, with the exception of the surface Ω, corresponds to the solution of this problem. On passing through Ω the function $u(x)$ varies by a jump since the crack edges in the field $\sigma_0(x)$ are displaced. (For external fields in which the relative displacement in the crack edges is lacking, the problem has the trivial solution $\sigma(x) = \sigma_0(x)$).

Let us also note that in the absence of mass forces the stress tensor $\sigma(x)$ in a medium with a crack will satisfy the equation $\operatorname{div}\sigma(x) = 0$ in the whole space including the surface Ω also.

The presence of a finite jump in the displacement field $u(x)$ on Ω as well as the mentioned property of the stress tensor permit interpreting the crack as a Somigliana dislocation induced by the external field /1,2/. As is known, the latter is a slit in an elastic medium whose edges are displaced by a given vector $b(x)$. The cavities being formed here fill the material of the initial medium (or remove the excess material), execute juncture of all the surfaces making contact, and then remove the forces which displaced the slit edges. In the case of a crack, the vector $b(x)$ is not known in advance and should be selected so that the total stress tensor (external and internal) would satisfy the given boundary conditions on the edges of the slit.

Let us examine the fundamental corollaries of the analogy mentioned. The singular density of the dislocation moments $m(x)$ which corresponds to a Somigliana dislocation, has the form /3/

$$m_{\alpha\beta}(x) = n_\alpha(x)b_\beta(x)\delta(\Omega) \tag{1.2}$$

Here $\delta(\Omega)$ is a delta function concentrated on the surface Ω and $b(x) = u^+(x) - u^-(x)$ is the jump in the displacement vector on Ω which agrees in the case of a crack with the vector of its opening, the plus sign denotes the limit value of the vector u on Ω from the normal side, and the minus sign from the opposite side.

The stress field in a medium with a crack can now be represented in the form of a sum of the external field $\sigma_0(x)$ and the internal stresses due to the dislocation moments of the

*Prikl.Matem.Mekhan.,45,No.2,361-370,1981 density (1.2). The strain tensor $\varepsilon(x)$ also allows of an analogous representation. Using the results of the continuum theory of dislocations /3/, we have

$$\varepsilon_{\alpha\beta}(x)=\varepsilon_{0\alpha\beta}(x)+\int_\Omega U^{\lambda\mu}_{\alpha\beta}(x-x')\,n_\mu(x')\,b_\lambda(x')\,d\Omega' \tag{1.3}$$

$$\sigma^{\alpha\beta}(x)=\sigma_0^{\alpha\beta}(x)+\int_\Omega S^{\alpha\beta\lambda\mu}(x-x')\,n_\mu(x')\,b_\lambda(x')\,d\Omega' \tag{1.4}$$

The kernel $U(x)$ of the integral operator in (1.3) is expressed in terms of the Green's tensor for the displacement $G(x)$ of the initial (homogeneous) medium with the elastic modulus tensor by the formula $c^{\alpha\beta\lambda\mu}$

$$U^{\lambda\mu}_{\alpha\beta}(x)=-[\nabla_\alpha\nabla_\rho G_{\beta\nu}c^{\rho\nu\lambda\mu}]_{(\alpha\beta)} \tag{1.5}$$

where ∇, the gradient operator in $\mathbf{R}^3$, is a three-dimensional Euclidean space and the parentheses denote symmetrization in the corresponding subscripts.

In turn, the kernel $S(x)$ of the integral operator in (1.4) is expressed in terms of the Green's tensor for the internal stresses $Z(x)$ of the initial medium ($e^{\alpha\beta\lambda}$ is the Levi-Civitta symbol)

$$S^{\alpha\beta\lambda\mu}(x)=\operatorname{rot}_\nu{}^\lambda\operatorname{rot}_\rho{}^\mu Z^{\alpha\beta\nu\rho}(x),\quad \operatorname{rot}^{\alpha\beta}=e^{\alpha\lambda\beta}\nabla_\lambda \tag{1.6}$$

It can be shown /3/ that the tensors U and S are connected by the relationship ($\delta(x)$ is the Dirac delta function)

$$S^{\alpha\beta\lambda\mu}(x)=c^{\alpha\beta\nu\rho}U^{\lambda\mu}_{\nu\rho}(x)-c^{\alpha\beta\lambda\mu}\delta(x) \tag{1.7}$$

Let us note that the tensor ε defined by the relationship (1.3) equals the sum of the elastic $c^{-1}\sigma$ and "plastic" $m_{(\alpha\beta)}$ components, where σ and m have the form (1.4) and (1.2), respectively.

The Green's tensor for the displacement $G(x)$ satisfies the equation

$$\nabla_\alpha c^{\alpha\beta\lambda\mu}\nabla_\lambda G_{\mu\nu}(x)=-\delta_\nu{}^\beta\delta(x) \tag{1.8}$$

where $\delta_\beta{}^\alpha$ is the Kronecker symbol, and the Green's tensor for the internal stresses $Z(x)$ satisfies the system of second order partial differential equations presented in /3/. The explicit expressions for $G(x)$ and $Z(x)$ are known only in particular cases for symmetry of the medium /4/. In the general case $G(x)$ and $Z(x)$ are even homogeneous functions of degree -1. It then follows from (1.5) and (1.6) that $U(x)$ and $S(x)$ are even homogeneous generalized functions of degree -3.

Let us note that it is possible to arrive at relationships of the form (1.3) and (1.4) by representing the solution of the elastic problem $u(x)$ in the form of a sum of the vector potential of the external field $u_0(x)$ and the potential of a double layer with the density $b(x)$ lumped in Ω (see /5/, for instance). The expression for $\varepsilon(x)$ will hence agree with (1.3), and the expression for $\sigma(x)$ with (1.4) if the first term in the right side of (1.7) is taken as the kernel $S(x)$. The stress field obtained in such a manner will differ from $\sigma(x)$ in the form (1.4) only by the singular component $-c^{\alpha\beta\lambda\mu}n_\mu(x)b_\lambda(x)\delta(\Omega)$, lumped on the crack surface Ω.

The mentioned difference is due to the fact that in selecting the solution in the form of a double layer potential, the crack is not modeled by dislocation but by force singularities, by a certain distribution of force dipoles in Ω /6/. The appropriate field $\sigma(x)$ hence contains a singular component lumped in Ω and satisfies the equation

$$\nabla_\alpha\sigma^{\alpha\beta}(x)=\nabla_\alpha q^{\alpha\beta}(x),\quad q^{\alpha\beta}(x)=c^{\alpha\beta\lambda\mu}n_\mu(x)\,b_\lambda(x)\,\delta(\Omega)$$

where $q(x)$ is the singular moment density of the force dipoles modeling the crack.

Let us note that the tensor $\sigma(x)$ of the form (1.4) satisfies the equation $\operatorname{div}\sigma(x)=0$ in all space, as follows from the representation (1.6). If Ω is a Liapunov surface, and the denisty $b(x)$ is twice differentiable on Ω, then the vector of the forces $f^\alpha(x)=n_\beta(x)\sigma^{\alpha\beta}(x)$, corresponding to the stresses (1.4), will be a continuous bounded function in all space with the exception, perhaps, of the contour Γ, the boundary of Ω /1,2/. (Here $n(x)$ is understood to be an arbitrary smooth continuation of the field of the normal given on Ω, in $\mathbf{R}^3$).

Let us now write the equation for the vector field $b(x)$ on Ω. From the relationship (1.4) and the boundary condition (1.1) we have

$$(T^{\alpha\beta}b_\beta)(x)=\int_\Omega T^{\alpha\beta}(x,x')\,b_\beta(x')\,d\Omega'=n_\beta(x)\sigma_0^{\alpha\beta}(x),\quad x\in\Omega \tag{1.9}$$

$$T^{\alpha\beta}(x, x') = -n_\lambda(x) S^{\lambda\alpha\beta\mu}(x - x') n_\mu(x') \tag{1.10}$$

The stresses and strains in a medium with a crack are reproduced uniquely from the relationships (1.3) and (1.4) in the vector field $b(x)$.

Let us note that the operator $\mathbf{T}$ in (1.9) can be written in the form of an integral operator with the kernel $T(x, x')$ only provisionally since the corresponding integral diverges formally for $x \in \Omega$ for arbitrarily smooth $b(x)$ ($T(x, x') \sim |x - x'|^{-3}$ as $x' \to x$).

Let us turn to an investigation of the properties of the operator $\mathbf{T}$ and the construction of the regularization formula for the integral in (1.9).

2. The generalized function $T(x)$. We start with the case when Ω is a plane in $\mathbf{R}^3$ with the equation $x_3 = 0$ (x_1, x_2, x_3 are Cartesian coordinates in $\mathbf{R}^3$), and $b(x_1, x_2)$ is a function of the class $S(\mathbf{R}^2)$ in Ω /7/. Here $n = \mathrm{const}$ and the kernel $T(x, x')$ in (1.9) depends only on the difference of the arguments $x - x'$. Therefore, $\mathbf{T}$ is the convolution operator with the generalized function $T(x)$ acting on the fundamental functions from $S(\mathbf{R}^2)$.

Let us note that the function $T(x) = T(x_1, x_2)$ is generated by the generalized function $S(x_1, x_2, x_3)$ defined by the relationship (1.6) (or (1.7)), and acting on the fundamental functions in $\mathbf{R}^3$. From (1.10) we have

$$T^{\alpha\beta}(x_1, x_2) = -S^{3\alpha\beta 3}(x_1, x_2, x_3)|_{x_3=0} \tag{2.1}$$

It hence follows that the Fourier transform of the function $T(x_1, x_2)$ has the form

$$T^{\alpha\beta}(k_1, k_2) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} S^{3\alpha\beta 3}(k_1, k_2, k_3)\, dk_3 \tag{2.2}$$

Here and below, the Fourier transform of a function has the argument k and its x-representation the argument x.

From (1.7), (1.5) and (1.8) we have

$$\begin{aligned} &S^{\alpha\beta\lambda\mu}(k_1, k_2, k_3) = c^{\alpha\beta\nu\rho} k_\rho G_{\nu\tau}(k_1, k_2, k_3) k_\sigma c^{\sigma\tau\lambda\mu} - c^{\alpha\beta\lambda\mu} \\ &G(k) = L^{-1}(k),\ L^{\alpha\beta}(k_1, k_2, k_3) = c^{\alpha\beta\lambda\mu} k_\lambda k_\mu \end{aligned} \tag{2.3}$$

By using these relationships it can be established that the integral in (2.2) converges absolutely and defines the even homogeneous function $T(k)$ of the first degree in k (k_1, k_2).

The action of the generalized function $T(x)$ on any fundamental function $\varphi(x) \in S(\mathbf{R}^2)$ can now be determined by the relationship

$$(T, \varphi) = \iint_{-\infty}^{\infty} T(x_1, x_2)\varphi(x_1, x_2)\, dx_1\, dx_2 = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} T(k_1, k_2)\varphi(k_1, k_2)\, dk_1\, dk_2 \tag{2.4}$$

which follows from the Parseval formula. The last integral converges absolutely.

Let us obtain a regularized expression for (T, φ) by using the formal expression of the generalized function $T(x)$ in x-space. We note first that $T(k)$ can always be represented in the form

$$T^{\alpha\beta}(k) = -Q^{\alpha\beta\lambda\mu}(k) k_\lambda k_\mu,\ Q^{\alpha\beta\lambda\mu}(k) = -T^{\alpha\beta}(k) k^\lambda k^\mu |k|^{-4}, \quad \lambda, \mu = 1, 2 \tag{2.5}$$

where $k = k(k_1, k_2)$, and $Q(k)$ is an even homogeneous function of degree -1. Therefore, there exists a $Q(x)$, an even homogeneous function of degree -1. This function is integrable at zero and because of (2.5) is related to $T(x)$ by means of the relationship

$$T^{\alpha\beta}(x_1, x_2) = \nabla_\lambda \nabla_\mu Q^{\alpha\beta\lambda\mu}(x_1, x_2), \quad \lambda, \mu = 1, 2 \tag{2.6}$$

We use a scheme that is proposed /8/ for the construction of the regularization of generalized functions of the type of derivatives of homogeneous regular functionals. Let ω be a domain in $\mathbf{R}^2$ with a smooth boundary γ, for which the point $x = 0$ where the function $T(x)$ is not integrable, is an interior point, while $\bar{\omega}$ is the complement of ω in all space.

By definition of the derivative of a generalized function, we have for any fundamental $\varphi(x)$ (subscripts have been omitted for simplicity)

$$(\nabla\nabla Q, \varphi) = -(\nabla Q, \nabla\varphi) = -\int_{\bar{\omega}} \nabla Q(x)\nabla\varphi(x)\, dx - \int_{\omega} \nabla Q(x)[\nabla\varphi(x) - \nabla\varphi(0)]\, dx - \oint_{\gamma} Q(x)\nu(x)\, d\gamma \nabla\varphi(0)$$

It is taken into account here that $\nabla Q(x)$ is a homogeneous function of degree -2 in $\mathbf{R}^2$, and the representation presented in /8/ is used for its regularization, $\nu(x)$ is the external normal to the contour γ, ∇ is the gradient in $\mathbf{R}^2$.

Now, transferring the derivative in the first integral into ∇Q, we obtain by taking account of (2.6)

$$(T,\varphi)=\int_{\bar{\omega}} T(x)[\varphi(x)-\varphi(0)]\,dx-\int_{\omega}\nabla Q(x)[\nabla\varphi(x)-\nabla\varphi(0)]\,dx+$$
$$\oint_{\gamma}\nabla Q(x)[\varphi(x)-\varphi(0)]\,\nu(x)\,d\gamma-\oint_{\gamma}Q(x)\,\nu(x)\,d\gamma\nabla\varphi(0)$$

Let ω be a circular domain of radius ρ. If $\rho\to 0$, then the second integral in this relationship will vanish since $Q(x)\sim|x|^{-1}$, the contour integrals over γ vanish because of the evenness of $Q(x)$, and the first integral tends to an integral in the Cauchy principal value sense and exists because of the definiteness of (T,φ). Therefore, the regularization of the generalized function $T(x)$ takes the form

$$(T,\varphi)=⨍ T(x)[\varphi(x)-\varphi(0)]dx \tag{2.7}$$

where the bar denotes an integral in the Cauchy principal value sense, and integration here is over the whole plane $\mathbf{R}^2$.

Let us note that the scheme elucidated is extended to the case of a space of any dimensionality $n\geqslant 1$, and the regularization of (2.7) holds for every homogeneous generalized function in $\mathbf{R}^n$, whose Fourier transform is an even homogeneous function of degree 1.

3. Regular representation of the operator T by functions from $C^\infty(\Omega)$.

As before, let Ω be the plane $x_3=0$. The convolution operator with the generalized function $T(x)$ in (1.9) can be determined in functions from $S(\mathbf{R}^2)$ by means of the formula

$$(\mathbf{T}b)(x_1,x_2)=\frac{1}{(2\pi)^3}\iint_{-\infty}^{\infty}T(k_1,k_2)\,b(k_1,k_2)\exp[-i(k_1x_1+k_2x_2)]\,dk_1\,dk_2 \tag{3.1}$$

Here $T(k_1,k_2)$ has the form (2.2), and the integral is here absolutely convergent $(b\in S(\mathbf{R}^2))$.

The representation (3.1) shows that $\mathbf{T}$ is a pseudo-differential operator with the symbol $T(k_1,k_2)$ /7/.

We obtain another formula for the regular representation of the operator $\mathbf{T}$ in functions from $S(\mathbf{R}^2)$ by using the relationships (2.7)

$$(Tb)(x)=⨍ T(x-x')[b(x')-b(x)]dx' \tag{3.2}$$

Here the integral is evaluated over the whole plane Ω, where continuous differentiability and boundedness of $b(x)$ in Ω for any $x\in\Omega$ are sufficient for its existence.

Now, let us turn to the case when Ω is a smooth simply connected surface in $\mathbf{R}^3$, bounded by the contour Γ, and $b(x)\in C^\infty(\Omega)$. Let us note that the three-dimensional Fourier transform of the function $S(x)$, which generates the kernel $T(x,x')$ of the operator $\mathbf{T}$, has the form (2.2) and is a homogeneous function of zero degree in k. The pseudo-differential operator with symbol $S(k)$ allows the following regular representation in functions from $S(\mathbf{R}^3)$ (D is a known constant) /7/:

$$(S\psi)(x)=\int S(x-x')\psi(x')dx'=⨍ S(x-x')\psi(x')dx'+D\psi(x),\ \psi(x)\in S(\mathbf{R}^3) \tag{3.3}$$

Let $h_i(x)$ be a sequence of functions from $S(\mathbf{R}^3)$ that converges to $\delta(\Omega)$ as $i\to\infty$. We predefine $n(x)$ and $b(x)$ given on Ω in the whole space $\mathbf{R}^3$ by using an arbitrary smooth continuation. Then the action of an operator $\mathbf{T}$ on any $b(x)\in C^\infty(\Omega)$ can be determined by the formula

$$(\mathbf{T}b)(x)=-\lim_{i\to\infty}n(x)\int S(x-x')n(x')b(x')h_i(x')dx',\quad x\in\Omega \tag{3.4}$$

where the integral over $\mathbf{R}^3$ is understood in the sense of the regularization of (3.3).

Furthermore, taking account of (1.6) for $S(x)$ and the Stokes theorem, we have the equality

$$\int_{\Omega}S^{\alpha\beta\lambda\mu}(x-x')\,n_\mu(x')\,d\Omega'=\oint_{\Gamma}\operatorname{rot}_\nu{}^{\lambda}Z^{\alpha\beta\nu\mu}(x-x')\,d\Gamma_\mu' \tag{3.5}$$

where $d\Gamma_\mu$ is a vector element of length on the contour Γ, whose orientation is matched to the orientation of Ω by the usual rule.

Using (3.4) and (3.5), we represent $(\mathbf{T}b)(x)$ in the form

$$(\mathbf{T}b)(x)=-\lim_{i\to\infty}n(x)\int S(x-x')\,n(x')\,[b(x')-b(x)]\,h_i(x')\,dx'+n(x)\oint\Pi(x-x')\,d\Gamma'b(x),\quad x\in\Omega$$
$$\Pi^{\alpha\beta\lambda\mu}(x)=-\operatorname{rot}_\nu{}^{\lambda}Z^{\alpha\beta\nu\mu}(x)$$

Taking account of the regularization of (3.3) and passing to the limit as $i\to\infty$, we obtain the final formula for the representation of the operator $\mathbf{T}$ by functions from $C^\infty(\Omega)$

$$(T^{\alpha\beta}b_\beta)(x)=\int_\Omega T^{\alpha\beta}(x,x')[b_\beta(x')-b_\beta(x)]\,d\Omega'+n_\nu(x)\oint_\Gamma \Pi^{\nu\alpha\beta\lambda}(x-x')\,d\Gamma_\beta' b_\lambda(x),\quad x\in\Omega \tag{3.6}$$

Here the first integral in Ω is understood in the Cauchy principal value sense, and exists because of the existence of the integral in (3.2).

Let us note that it is sufficient for the existence of the integrals in (3.6) that the function $b(x)$ be continuously differentiable on Ω and vanish on Γ. By continuity, the operator $\mathbf{T}$ is continued to an unbounded operator from $H_s(\Omega)$ into $H_{s-1}(\Omega)$ and is a generalized pseudo-differential operator with a principal homogeneous symbol, a homogeneous function of degree one.

The equation

$$(\mathbf{T}b)(x)=f(x),\quad x\in\Omega \tag{3.7}$$

has the unique solution $b\in \overset{\circ}{H}_{\delta+1/2}(\Omega)$ for $f\in H_{\delta-1/2}(\Omega)$ where $|\delta|<1/2$ is arbitrary. (The definition of Sobolev—Slobodetskii functional spaces $H_s(\Omega)$ and $H_s^\circ(\Omega)$ is found in /7/). The appropriate theorem is proved in /7/.

For a function $f(x)$ infinitely differentiable in Ω, the asymptotic of the solution of (3.7) near a smooth boundary, the contour Γ, has the form /7/

$$b(x)=\beta(x_0)\sqrt{r}+O(r^{3/2}) \tag{3.8}$$

where r is the distance from the point $x\in\Omega$ to $x_0\in\Gamma$ along the normal to Γ, and $\beta(x_0)$ is a function infinitely differentiable along Γ.

Now, we consider $\mathbf{T}$ as an operator in the Hilbert space $L_2(\Omega)\equiv H_0(\Omega)$. We consider the domain of definition $\mathbf{T}$ functions from the space $C_0^\infty(\Omega)$ compact in $L_2(\Omega)$ of finite infinitely differentiable functions in Ω, whose carrier is concentrated in internal subdomains of Ω. It can be shown that $\mathbf{T}$ is a positive operator.

Let $u_1(x)$ be the displacement vector, and $\sigma_1(x)$ the internal stress tensor in a medium with dislocation moments of density $n_\alpha(x)b_\beta(x)$ $(n\in C^\infty(\Omega),\ b\in C_0^\infty(\Omega))$ distributed on the surface Ω. It follows from (1.4) and (1.10) that $(\mathbf{T}b)(x)$ is the value of the vector $-n_\alpha(x)\sigma_1^{\alpha\beta}(x)$ on Ω.

Let Ω_+ denote the positive, and Ω_- the negative side of the surface Ω, whose selection is determined by the orientation of the normal n. Let us consider the integral

$$I(b)=\int_\Omega (T^{\alpha\beta}b_\beta)\,b_\alpha\,d\Omega=-\int_{\Omega_+} n_\alpha^+(x)\,\sigma_1^{\alpha\beta}(x)\,u_{1\beta}^+(x)\,d\Omega-\int_{\Omega_-} n_\alpha^-(x)\,\sigma_1^{\alpha\beta}(x)\,u_{1\beta}^-(x)\,d\Omega;\quad n^+=n,\ n^-=-n$$

where it is taken into account that $b(x)=u_1^+(x)-u_1^-(x)$.

Let us apply the Ostrogradskii formula to the right side. Taking into account that $\operatorname{div}\sigma_1(x)=0$, we obtain

$$I(b)=\int_{R^3\setminus\Omega}\nabla_\alpha[\sigma_1^{\alpha\beta}(x)\,u_{1\beta}(x)]\,dx=\int_{R^3\setminus\Omega}\varepsilon_{1\alpha\beta}(x)\,c^{\alpha\beta\lambda\mu}\varepsilon_{1\lambda\mu}(x)\,dx\geqslant 0;\quad \varepsilon_{1\alpha\beta}(x)=\nabla_{(\alpha}u_{1\beta)}(x) \tag{3.9}$$

Here we used the equality $\sigma_1^{\alpha\beta}=c^{\alpha\beta\lambda\mu}\varepsilon_{1\lambda\mu}$, which is valid outside Ω, the integral over $R^3\setminus\Omega$ converges since $\varepsilon_1(x)\sim|x|^{-3}$ at infinity, and the field $\varepsilon_1(x)$ is bounded in the neighborhood of Ω for $b\in C_0^\infty(\Omega)$. The last inequality in (3.9) is a consequence of the positive definiteness of the tensor $c^{\alpha\beta\lambda\mu}$.

Therefore $I(b)=(\mathbf{T}b,b)\geqslant 0$, where equality is achieved only for $b=0$. Therefore, $\mathbf{T}$ is a positive operator which is moreover symmetric, as can be verified by using (1.10).

The property obtained for the operator $\mathbf{T}$ permits the assertion that the solution of (3.7) yields the minimum of the functional

$$F(b)=\int_\Omega (T^{\alpha\beta}b_\beta)\,b_\alpha\,d\Omega-2\int_\Omega f^\alpha b_\alpha\,d\Omega$$

and, therefore, direct variational methods /9/ can be used for an approximate evaluation of $b(x)$.

Remark on the numerical solution of (3.7). If the explicit expression for the function $T(x,x')$ is known, then the scheme, examined in /5/, say, can be used for the solution of (3.7). Let us partition the surface Ω into N subdomains Ω_i. We approximate the function $b(x)$ within each subdomain Ω_i by a linear combination of standard functions with unknown coefficients. Substituting $b(x)$ in such form into (3.7), we obtain a system of linear equations in the constant coefficients of the approximation.

In particular, if $b(x) = \text{const}$ for $x \in \Omega_i$, then the system to determine the vectors b^i ($i = 1, 2, \ldots, N$), the values of $b(x)$ within Ω_i, takes the form

$$\sum_{j=1}^{N} T_{ij}^{\alpha\beta} b_\beta^{\,j} = f_i^{\,\alpha} \tag{3.10}$$

where $f_i = f(x_i)$ is the value of the right side of (3.7) at a nodal (interior) point x_i of the domain Ω_i, and the tensors T_{ij} are determined by the relationship

$$T_{ij} = \int_{\Omega_j} T(x_i, x')\, d\Omega'$$

For $i \neq j$ the integrand has no singularities in Ω_j, and this integral can be evaluated by any approximate method.

If $i = j$, the preceding formula becomes meaningless, and the regularization (3.6) should be used to evaluate the elements T_{ii}. Since $b = \text{const}$ in Ω_j, the first term on the right in (3.6) vanishes and

$$T_{ii}^{\alpha\beta} = -\oint_{\Gamma_i} n_\lambda(x_i)\, \mathrm{rot}_\mu{}^\beta Z^{\lambda\alpha\mu\nu}(x_i - x')\, d\Gamma_\nu{}'$$

where Γ_i is the boundary contour of the domain Ω_i, $x_i \overline{\in} \Gamma_i$.

If the contour Γ_i lies entirely in one plane ω_i, then the relation (3.2) can be used instead of the regularization (3.6). Then T_{ii} is represented in the form of an absolutely convergent integral

$$T_{ii} = -\int_{\bar{\omega}_i} T(x_i, x')\, d\Omega'$$

where $\bar{\omega}_i$ is the part of the plane ω_i outside the contour Γ_i. The formulas presented here facilitate realization of the method of solution under consideration substantially. Tedious schemes to evaluate the elements T_{ii} were proposed in /5,10,11/, and examples of the solution of the system (3.10) are given there.

4. Tensorial stress intensity factor on the crack contour. Let us consider the asymptotic of the stress field $\sigma(x)$ outside the crack in the neighborhood of its edge Γ. Let y_1, y_2, y_3 be local Cartesian coordinates at the point $x_0 \in \Gamma$, where the y_3 axis is directed along the limit normal to Ω at the point x_0, the y_2 axis is along the tangent to Γ, then the y_1 axis lies in the tangent plane to Ω at the point x_0. Taking account of the asymptotic (3.8), we have an asymptotic of the vector b in the neighborhood of x_0

$$b(y) = \beta(x_0)\sqrt{y_1} + O(y_1^{3/2})$$

Using (1.4), we write an expression for σ at the point $z = (-r\cos\theta,\ 0,\ -r\sin\theta)$, where r is the spacing between the point z and the origin of the y_i coordinate system, and θ is the polar angle in the plane (y_1, y_3)

$$\sigma(z) = \sigma_0(z) + \frac{1}{\sqrt{r}} \int_{\Omega(r)} S(\cos\theta + \xi_1, \xi_2, \sin\theta + \xi_3) \times n(r\xi)\beta(x_0)\sqrt{\xi_1}\, d\Omega_\xi + O(\sqrt{r}), \quad \xi_i = r^{-1} y_i \tag{4.1}$$

It is taken into account here that $S(x)$ is homogeneous, of degree -3, an even function. It can be shown that as $r \to 0$ the integral tends to a finite limit, and the stresses therefore have the singularity $r^{-1/2}$.

Let us consider the tensor function $J(\theta, x_0)$ (the tensorial stress intensity factor), which is of interest for applications and is defined by the relationship

$$J(\theta, x_0) = \lim \sqrt{r}\,\sigma(z), \quad r \to 0$$

It follows from (4.1) that

$$\sigma(z) = \frac{1}{\sqrt{r}} J(\theta, x_0) + O(1)$$

and the components of the tensor J have the form

$$J^{\alpha\beta}(\theta, x_0) = s^{\alpha\beta\lambda\mu}(\theta)\, n_\lambda(x_0)\, \beta_\mu(x_0) \tag{4.2}$$

$$s(\theta) = \int_0^\infty \sqrt{\xi_1}\, d\xi_1 \int_{-\infty}^{\infty} S(\cos\theta + \xi_1, \xi_2, \sin\theta)\, d\xi_2 \tag{4.3}$$

where $n(x_0)$ is the limit value of the normal to Ω at the point $x_0 \in \Gamma$.

It is hence seen that the function $J(\theta, x_0)$ is representable in the form of two factors, the first of which $s(\theta)n(x_0)$ is independent of the shape of the surface Ω and the external field applied to the medium, and is determined by the local orientation of the axes y_1, y_2, y_3 at the point $x_0 \in \Gamma$. The second factor, the vector $\beta(x_0)$, is a functional of the whole surface Ω and the external field $\sigma_0(x)$.

The function $J(\theta, x_0)$ allows of graphical interpretation if it is taken into account that

$$\int_{-\infty}^{\infty} S(\xi_1, \xi_2, \xi_3)\, d\xi_2$$

is substantially an analog of the tensor $S(x)$ in the plane problem of deformation and complex shear (in the dimensionless coordinates ξ_i) of a homogeneous medium with the moduli $c^{\alpha\beta\lambda\mu}$, where the normal to the plane of deformation (ξ_1, ξ_3) is directed along the ξ_2 axis. The tensor $J(\theta, x_0)$ here agrees with the stress tensor at the point $\xi_1 = -\cos\theta$, $\xi_3 = -\sin\theta$, when a jump in the displacement vector which varies according to the law $\beta(x_0)\sqrt{\xi_1}$ is given along the positive ξ_1 semi-axis.

Let us note that the tensor $S(x)$ is expressed for the plane case in terms of the Green's function $G(x)$ of the plane problem by means of formulas analogous to (1.5) and (1.7).

In the plane case, the explicit expression is known for the function $G(x)$ for arbitrary anisotropy of the tensor of the elastic constants, hence, construction of a tensor $s(\theta)$ of the form (4.3) reduces to evaluation of the standard integrals, and its expression can also be found explicitly.

The tensor J can be represented in the form of the sum of three tensors corresponding to three components of the vector $\beta(x_0)$ in the axes y_1, y_2, y_3

$$J = J_1 + J_2 + J_3;\quad J_i^{\alpha\beta}(\theta, x_0) = s^{\alpha\beta\lambda i}(\theta) n_\lambda(x_0) \beta_i(x_0) \tag{4.4}$$

(no summation over i!).

The tensors $s^{\alpha\beta\lambda 1}(\theta)$ and $s^{\alpha\beta\lambda 3}(\theta)$ are found from the solution of the corresponding plane problem, and $s^{\alpha\beta\lambda 2}(\theta)$ from the solution of the antiplane problem (complex shear).

Let us note that the asymptotic of the stress field in the neighborhood of a crack edge is usually characterized by the stress intensity factors $K_{\mathrm{I}}, K_{\mathrm{II}}, K_{\mathrm{III}}$ in the theory of elasticity and fracture mechanics (see /12/, for instance). By using the definition of these coefficients, it can be shown that their relation to the tensor components $J_i(\theta, x_0)$ is given by

$$K_{\mathrm{I}}(x_0) = \sqrt{2\pi} J_3^{33}(0, x_0),\quad K_{\mathrm{II}}(x_0) = \sqrt{2\pi} J_1^{13}(0, x_0),\quad K_{\mathrm{III}}(x_0) = \sqrt{2\pi} J_2^{23}(0, x_0)$$

From here and (4.4) it follows that to the accuracy of constant factors dependent on the elastic constants, the stress intensity factors agree with the components of the vector $\beta(x_0)$.

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